

SEMIINFINITE COHOMOLOGY OF LIE-* ALGEBRAS

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1. INTRODUCTION.

This paper is a natural extension of the previous note [Ar2]. Semiinfinite cohomology of Tate Lie algebra was defined in that note in terms of some duality resembling Koszul duality. The language of differential graded Lie algebroids was the main technical tool of the note.

The present note is devoted to globalization of the main construction from [Ar2] in the following sense. The setup in [Ar2] included a suitably chosen module over a Tate Lie algebra \mathfrak{g} with a fixed Lie subalgebra \mathfrak{b} being a c -lattice in \mathfrak{g} .

The rough global analogue of this picture is as follows. Consider a compact curve X over a field of characteristic zero. Denote by $\text{mod-}D_X$ the category of (right) D -modules on X . We fix a Lie algebra \mathcal{G} in the category $\text{mod-}D_X$. This data can be viewed as a family of Lie algebras \mathfrak{g}_x along the curve X . Another part of the data includes a Lie subalgebra $\mathcal{B} \subset \mathcal{G}$. So the problem is to define semiinfinite cohomology complex of such pair.

In fact we need some additional constraints on the pair $\mathcal{B} \subset \mathcal{G}$. So the formal picture starts from a different notion of a Lie-* algebra \mathcal{L} on X (see the precise definition in Section 2). Roughly speaking a D_X -locally free Lie-* algebra is a D -module incarnation of a Lie algebra in the category of vector bundles on X with the bracket given by a differential operator. We define two types of modules over a Lie-* algebra (see 2.2.1). The first one called a *Lie-* module* is just a D -module incarnation of the module over the Lie algebra in the category of vector bundles, like above, with the action given by a differential operator. Still we will be more interested in the second type of modules over a Lie-* algebra called *chiral modules* (see 2.2.1 for the definition).

So starting from a Lie-* algebra \mathcal{L} and a chiral module \mathcal{M} we perform the main construction more or less parallel to the one from [Ar2]. Namely we define the Lie algebra $\mathcal{G} = \mathcal{G}(\mathcal{L})$ in the category $\text{mod-}D_X$ with the Lie subalgebra $\mathcal{B} = \mathcal{B}(\mathcal{L}) \subset \mathcal{G}(\mathcal{L})$. We show that a \mathcal{L} -chiral module \mathcal{M} becomes a $\mathcal{G}(\mathcal{L})$ -module.

Next, imitating the construction of [Ar2] Section 4, we define a DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L})$ in the category of D_X -modules over a DG \otimes -algebra $\mathcal{R}^\bullet = \mathcal{R}^\bullet(\mathcal{L})$. Koszul duality type construction provides a *left* DG-module $C^\bullet(\mathcal{L}, \mathcal{M})$ over $\mathcal{A}^\bullet(\mathcal{L})$.

To go further one needs to pass to a central extension of \mathcal{L} called the Tate central extension and denoted by $\mathcal{L}_{\text{Tate}}$. Let \mathcal{M} be a chiral module over $\mathcal{L}_{\text{Tate}}$. It turns out that the complex of D -modules $C^\bullet(\mathcal{L}_{\text{Tate}}, \mathcal{M})$ by some antipode construction becomes a *right* module over $\mathcal{A}^\bullet(\mathcal{L})$.

Finally we consider the homological Chevalley complex of the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L})$ in the category of D_X -modules with coefficients in $C^\bullet(\mathcal{L}_{\text{Tate}}, \mathcal{M})$ (see 4.2 for the definition of the homological Chevalley complex of a DG Lie algebroid). We call

the obtained complex of D-modules the global standard semiinfinite complex of the Lie-* algebra \mathcal{L} with coefficients in the chiral module \mathcal{M} .

Let us say a few words about the structure of the paper. In Section 2 we collect necessary definitions and simple facts about Lie-* algebras, Lie-* modules, chiral modules etc. Section 3 is devoted to the construction of the Tate central extension of a Lie-* algebra. Section 4 contains all the necessary constructions concerning DG Lie algebroids in the category of D_X -modules. In particular we present the definition of the homological Chevalley complex of a DG Lie algebroid with coefficients in a *right* DG-module. Section 5 is the heart of the paper. We present the constructions of the Lie algebras $\mathcal{B}(\mathcal{L})$ and $\mathcal{G}(\mathcal{L})$ in the category of D_X -modules. Then we define the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L})$ over the DG $\overset{!}{\otimes}$ -algebra $\mathcal{R}^\bullet(\mathcal{L})$. Finally after overcoming the problem of necessity to pass to the Tate central extension of \mathcal{L} we present the standard semiinfinite complex $C^{\frac{\infty}{2}}(\mathcal{L}_{\text{Tate}}, \mathcal{M})$ for a $\mathcal{L}_{\text{Tate}}$ -chiral module \mathcal{M} .

Note that the paper [BD] contains a construction of the global BRST complex for calculating the semiinfinite cohomology of a chiral module over a Lie-* algebra. Somehow the present paper grew out of an attempt to understand that construction avoiding the notions of chiral algebras, chiral enveloping algebras etc. The technique used in the definition in [BD] is quite different from ours and it is not checked that the two constructions give the same answer.

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2. LIE-* ALGEBRAS.

In this section we recall briefly basic notation and constructions concerning Lie-* algebras. In our exposition we follow [Ga]. We will be working over a fixed smooth curve X . Denote the diagonal embedding $X \hookrightarrow X \times X$ by Δ . The embedding of the complementary open set $X \times X \setminus X_\Delta \hookrightarrow X \times X$ is denoted by j .

Let D_X -mod (resp. mod- D_X) be the category of left (resp. right) modules \mathcal{M} over the sheaf of algebras of differential operators on X such that \mathcal{M} is quasicoherent over \mathcal{O}_X . It is known that the category D_X -mod is naturally a symmetric tensor category with the tensor product given by $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ for $\mathcal{M}, \mathcal{N} \in D_X$ -mod. Let $\Omega = \Omega_X^1$. Then the category mod- D_X becomes a symmetric tensor category with the tensor product given by

$$\mathcal{M} \overset{!}{\otimes} \mathcal{N} = (\mathcal{M} \otimes_{\mathcal{O}_X} \Omega^{-1}) \otimes_{\mathcal{O}_X} (\mathcal{N} \otimes_{\mathcal{O}_X} \Omega^{-1}) \otimes_{\mathcal{O}_X} \Omega.$$

2.1. Definition. Recall that a *Lie-* algebra* on X is a right D-module \mathcal{L} with the map

$$\{\cdot \boxtimes \cdot\} : \mathcal{L} \boxtimes \mathcal{L} \longrightarrow \Delta_!(\mathcal{L})$$

which is antisymmetric and satisfies the Jacobi identity in the following sense. If $a \boxtimes b \boxtimes c \cdot f(x, y, z)$ is a section of the $\mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{L}$ on $X \times X \times X$, then the element

$$\{\{f(x, y, z) \cdot a \boxtimes b\} \boxtimes c\} + \sigma_{1,2,3}(\{\{f(z, x, y) \cdot b \boxtimes c\} \boxtimes a\}) + \sigma_{1,2,3}(\{\{f(y, z, x) \cdot c \boxtimes a\} \boxtimes b\})$$

of $\Delta_{x=y=z!}(\mathcal{L})$ vanishes. Here $\sigma_{1,2,3}$ denotes the lift of the cyclic automorphism of $X \times X \times X: (x, y, z) \mapsto (y, z, x)$ to the D-module $\Delta_{x=y=z!}(\mathcal{L})$.

2.1.1. Note that if \mathcal{L} is a Lie- $*$ algebra, it follows from the definition that $DR^0(\mathcal{L})$ is a sheaf of ordinary Lie algebras; moreover it acts on \mathcal{L} by endomorphisms of the D-module structure that are derivations of the Lie- $*$ structure.

In particular, for an affine subset $U \subset X$, $DR^0(U, \mathcal{L})$ is a Lie algebra. Thus for any point $x \in X$ the topological spaces $DR^0(\text{Spec}(\hat{\mathcal{O}}_x), \mathcal{L})$ and $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L})$ carry the natural structures of topological Lie algebras.

2.1.2. Our next step consists of reformulation of the Lie- $*$ algebra definition in terms of Lie coalgebras in the standard tensor structure $\overset{!}{\otimes}$ on right D-modules.

Lemma: Let $\mathcal{M}_1, \mathcal{M}_2$ be two D-modules on X with \mathcal{M}_1 being locally free and finitely generated. Then:

(i) For a third D-module \mathcal{M} on X , there is a canonical isomorphism:

$$\text{Hom}_{D_X^2}(\mathcal{M} \boxtimes \mathcal{M}_1, \Delta_!(\mathcal{M}_2)) \xrightarrow{\sim} \text{Hom}_{D_X}(\mathcal{M}, \mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2).$$

(ii) The canonical map (from point (i)) $(\mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2) \boxtimes \mathcal{M}_1 \longrightarrow \Delta_!(\mathcal{M}_2)$ induces an

isomorphism $DR^0(\mathcal{M}_1^* \overset{!}{\otimes} \mathcal{M}_2) \longrightarrow \text{Hom}_D(\mathcal{M}_1, \mathcal{M}_2)$. \square

Below we always suppose that any Lie- $*$ algebra \mathcal{L} we work with is locally free and finitely generated as a D-module on X .

Corollary: For a Lie- $*$ algebra \mathcal{L} we have canonical maps

$$\text{co-ad} : \mathcal{L} \boxtimes \mathcal{L}^* \longrightarrow \Delta_!(\mathcal{L}^*) \text{ and } \text{co-br} : \mathcal{L}^* \rightarrow \mathcal{L}^* \overset{!}{\otimes} \mathcal{L}^*. \quad \square$$

We shall call the two maps of the above corollary “the co-adjoint action” and “the co-bracket”, respectively.

2.1.3. **Remark:** In particular note that to specify a structure of a Lie- $*$ algebra on a D-module \mathcal{L} is the same as to provide a structure of a *Lie coalgebra* (in the usual tensor structure on the category of right D-modules) on the D-module $\mathcal{L}^* := \underline{\text{Hom}}_{D_X}(\mathcal{L}, D_X \otimes \Omega_X)$. The co-bracket here is the one obtained in the previous Corollary.

2.2. **Modules over a Lie- $*$ algebra.** There are two different ways to define a notion of a module over a Lie- $*$ algebra.

2.2.1. A *Lie- $*$ module* over a Lie- $*$ algebra \mathcal{L} is a (right) D-module \mathcal{M} on X with a map

$$\rho : \mathcal{L} \boxtimes \mathcal{M} \rightarrow \Delta_!(\mathcal{M})$$

such that for a section $a \boxtimes b \boxtimes m$ of $\mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{M}$ the two sections

$$\rho(\{a \boxtimes b\}, m) \text{ and } \rho(a, \rho(b, m)) - \sigma_{1,2}(\rho(b, \rho(a, m)))$$

of $\Delta_{1,2,3!}(\mathcal{L})$ coincide.

Like in the Lie- $*$ algebra case a structure of the Lie- $*$ module on \mathcal{M} provides a structure of the sheaf of modules over the sheaf of Lie algebras $DR^0(\mathcal{L})$ on $DR^0(\mathcal{M})$. Moreover the sheaf of Lie algebras $DR^0(\mathcal{L})$ acts on \mathcal{M} and contrarywise such an

action recovers a Lie- * module structure if it is given by differential operators (see [BD], 2.5.4).

Remark: Similarly to 2.1.3 one can easily check that to specify a structure of a Lie- * module over the Lie- * algebra \mathcal{L} on a D-module \mathcal{M} is the same as to make \mathcal{M} a comodule over the Lie coalgebra \mathcal{L}^* , i.e. to provide a D-module map $\text{co-ac} : \mathcal{M} \longrightarrow \mathcal{L}^* \overset{!}{\otimes} \mathcal{M}$ satisfying certain coassociativity constraints.

2.2.2. A *chiral module* over a Lie- * algebra is again a (right) D-module \mathcal{M} on X , but with an operation

$$\rho : j_* j^*(\mathcal{L} \boxtimes \mathcal{M}) \rightarrow \Delta_!(\mathcal{M})$$

such that every section $f(x, y, z) \cdot a \boxtimes b \boxtimes m \in \Gamma(X \times X \times X \setminus (\Delta_{x=z} \cup \Delta_{y=z}), \mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{M})$ satisfies an identity as follows:

$$\rho(\{f(x, y, z) \cdot a \boxtimes b\}, m) = \rho(a, \rho(f(x, y, z) \cdot b, m)) - \sigma_{1,2}(\rho(b, \rho(f(y, x, z) \cdot a, m))),$$

as sections of $\Delta_{x=y=z!}(\mathcal{M})$.

2.3. Naive de Rham complex for a Lie- * module over a Lie- * algebra.

Let \mathcal{M} be a Lie- * module over a (locally D-free finitely generated) Lie- * algebra \mathcal{L} .

Consider a complex $C^\bullet(\mathcal{L}, \mathcal{M})$ of D-modules on X as follows: $C^k(\mathcal{L}, \mathcal{M}) = \Lambda^k(\mathcal{L}^*) \overset{!}{\otimes} \mathcal{M}$ as a D-module. The differential is given by

$$d(\omega_1 \wedge \dots \wedge \omega_k \otimes m) = \sum \omega_1 \wedge \dots \wedge \text{co-br}(\omega_i) \wedge \dots \wedge \omega_k \otimes m + \omega_1 \wedge \dots \wedge \omega_k \wedge \text{co-ac}(m).$$

Here ω_i are the sections of \mathcal{L}^* and m is the section of \mathcal{M} .

Lemma: The complex of D-modules $C^\bullet(\mathcal{L}, \mathcal{M})$ is well-defined, i.e. $d^2 = 0$. \square

Remark: We call the constructed complex the *naive de Rham complex* for the Lie- * module over the Lie- * algebra. However below we construct a more sophisticated complex for a *chiral module* over a Lie- * algebra. That complex is also of de Rham origin. It will be realized as de Rham complex of a certain DG Lie- * module over a certain *DG Lie- * algebroid*.

2.4. **De Rham DG-algebra of a Lie- * algebra.** Consider the complex $C^\bullet(\mathcal{L}, \mathcal{M})$ for \mathcal{M} equal to the trivial Lie- * module Ω over \mathcal{L} .

Lemma: The wedge product makes the complex $C^\bullet(\mathcal{L}, \Omega)$ into a supercommutative DG-algebra in the tensor category of right D-modules. \square

We denote this DG-algebra by $\mathcal{R}^\bullet = \mathcal{R}^\bullet(\mathcal{L})$.

3. TATE EXTENSION OF A LIE- * ALGEBRA.

3.1. **Matrix Lie- * algebra.** Let \mathcal{V} be a locally free finitely generated D-module on X , and let $\mathcal{V}^* := \underline{\text{Hom}}_{D_X}(\mathcal{V}, D_X \otimes \Omega_X)$ be its (Verdier) dual D-module. Consider the $\overset{!}{\otimes}$ -tensor product of \mathcal{V} and \mathcal{V}^* .

3.1.1. **Remark:** By Lemma 2.1.2(i) we have a Lie-* pairing

$$\langle \cdot \boxtimes \cdot \rangle : \mathcal{V} \boxtimes \mathcal{V}^* \longrightarrow \Delta_!(\Omega).$$

Lemma: The D-module $\mathcal{V} \overset{!}{\otimes} \mathcal{V}^* =: \mathcal{M}at$ carries a natural structure of an associative-* algebra on X .

Proof. The associative product map is defined as follows:

$$\begin{aligned} \text{as} : \left(\mathcal{V}_{(1)} \overset{!}{\otimes} \mathcal{V}_{(1)}^* \right) \boxtimes \left(\mathcal{V}_{(2)} \overset{!}{\otimes} \mathcal{V}_{(2)}^* \right) \\ \xrightarrow{\quad} \left((\mathcal{V}_{(1)} \boxtimes \Omega) \overset{!}{\otimes} (\Omega \boxtimes \mathcal{V}_{(2)}^*) \right) \overset{!}{\otimes} \left((\mathcal{V}_{(2)} \boxtimes \Omega) \overset{!}{\otimes} (\Omega \boxtimes \mathcal{V}_{(1)}^*) \right) \\ \xrightarrow{\langle \cdot \boxtimes \cdot \rangle \overset{!}{\otimes} \text{id}} \Delta_!(\Omega) \overset{!}{\otimes} \left((\mathcal{V}_{(2)} \boxtimes \Omega) \overset{!}{\otimes} (\Omega \boxtimes \mathcal{V}_{(1)}^*) \right) \xrightarrow{\quad} \Delta_!(\mathcal{V} \overset{!}{\otimes} \mathcal{V}^*). \end{aligned}$$

Here the indexes (1) and (2) denote the factors in the product, and the *-pairing is taken between the ones in the first pair of brackets. It is left to the reader to check the associativity of the product. \square

Now we obtain the Lie-* bracket on $\mathcal{M}at$ from the associative-* product in the usual way:

$$a \boxtimes b \mapsto \text{as}(a \boxtimes b) - \text{as}(b \boxtimes a).$$

We call $\mathcal{M}at$ the *matrix Lie-* algebra* of the D-module \mathcal{V} .

Note that the associative-* algebra $\mathcal{M}at$ acts both on \mathcal{V} and on \mathcal{V}^* in a canonical way and the pairing $\langle \cdot \boxtimes \cdot \rangle$ is $\mathcal{M}at$ -invariant.

3.2. **Tate extension of the Lie-* algebra $\mathcal{M}at$.** The material of this subsection is almost word to word copied from [BD], 2.6. We include it in our paper just for the sake of completeness.

3.2.1. The Tate extension is a canonical central extension of Lie-* algebras

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{M}at^b \xrightarrow{\pi^b} \mathcal{M}at \longrightarrow 0.$$

To define $\mathcal{M}at^b$ as a D-module consider the exact sequence of D-modules on $X \times X$

$$\mathcal{V} \boxtimes \mathcal{V}^* \xrightarrow{\varepsilon} j_* j^*(\mathcal{V} \boxtimes \mathcal{V}') \xrightarrow{\pi} \Delta_!(\mathcal{V} \overset{!}{\otimes} \mathcal{V}^*) \longrightarrow 0.$$

Here as before $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding, $j : U := X \times X \setminus \Delta(X) \hookrightarrow X \times X$ is the complementary embedding, and π is the canonical arrow.

Namely one has $\mathcal{V} \overset{!}{\otimes} \mathcal{V}^* = H^1 \Delta^!(\mathcal{V} \boxtimes \mathcal{V}^*) = \text{Coker } \varepsilon$; explicitly, π sends $(t_2 - t_1)^{-1} v \boxtimes v' \in j_* j^* \mathcal{V} \boxtimes \mathcal{V}^*$ to $v \otimes v' (dt)^{-1} \in \Delta_!(\mathcal{V} \overset{!}{\otimes} \mathcal{V}') \subset \Delta_!(\mathcal{V} \overset{!}{\otimes} \mathcal{V}^*)$. Note that $\langle \cdot \boxtimes \cdot \rangle$ vanishes on $\text{Ker } \varepsilon$ and, pushing forward the above exact sequence by $\langle \cdot \boxtimes \cdot \rangle$, we get an extension of $\Delta_!(\mathcal{V} \overset{!}{\otimes} \mathcal{V}^*)$ by $\Delta_!(\Omega)$.

This extension is supported on the diagonal. Applying $\Delta^!$ we get the Tate extension $\mathcal{M}at^b$. We denote the canonical morphism $j_* j^*(\mathcal{V} \boxtimes \mathcal{V}^*) \longrightarrow \Delta_! \mathcal{M}at^b$ as $\mu_{\mathcal{M}at}$.

3.2.2. The above construction is natural with respect to Lie-* algebras actions. Namely, assume that a Lie-* algebra \mathcal{L} acts on \mathcal{V} , \mathcal{V}^* so that $\langle \cdot \boxtimes \cdot \rangle$ is \mathcal{L} -invariant. Then the \mathcal{L} -action on $\mathcal{M}at = \mathcal{V} \overset{!}{\otimes} \mathcal{V}^*$ lifts canonically to an \mathcal{L} -action on $\mathcal{M}at^b$. To see this, consider the $D_X^{\otimes 2}$ -modules $\Delta(\mathcal{V} \boxtimes \mathcal{V}^*)$, $\Delta(j_* j^*(\mathcal{V} \boxtimes \mathcal{V}^*))$. The Lie algebra $h(\mathcal{L})$ acts on them in the obvious manner. The morphism $\langle \cdot \boxtimes \cdot \rangle : \Delta(\mathcal{V} \boxtimes \mathcal{V}^*) \longrightarrow \Delta(\Delta_!(\Omega))$ is $DR^0(\mathcal{L})$ -invariant. Therefore $DR^0(\mathcal{L})$ acts on $\mathcal{M}at^b$. This action is uniquely determined by property that $\mu_{\mathcal{M}at}$ is a morphism of $DR^0(\mathcal{M}at)$ -modules. Evidently the action of $DR^0(\mathcal{L})$ is given by differential operators so we have the desired \mathcal{L} -action on $\mathcal{M}at^b$.

In particular, the canonical $\mathcal{M}at$ -actions on \mathcal{V} , \mathcal{V}^* define a $\mathcal{M}at$ -action on $\mathcal{M}at^b$ that lifts the adjoint action on $\mathcal{M}at$. Composing this action with π^b we get an operation

$$\{ \cdot \boxtimes \cdot \} : \mathcal{M}at^b \boxtimes \mathcal{M}at^b \longrightarrow \Delta_!(\mathcal{M}at^b).$$

3.2.3. **Lemma:** This is a Lie-* bracket. □

3.3. **Tate extension of a Lie-* algebra.** Now let \mathcal{L} be an arbitrary D_X -locally free finitely generated Lie-* algebra. Recall that by Lemma 2.1.2 we have a canonical co-action map of the D_X -modules

$$\text{co-ac} : \mathcal{L} \longrightarrow \mathcal{L} \overset{!}{\otimes} \mathcal{L}^*.$$

We interpret as a (D-module) map $\text{can} : \mathcal{L} \longrightarrow \mathcal{M}at$.

Lemma: The map can is a morphism of Lie-* algebras. □

3.3.1. **Corollary:** For any D_X -locally free finitely generated Lie-* algebra \mathcal{L} there exists a central extension in the class of Lie-* algebras as follows

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{L}^b \longrightarrow \mathcal{L} \longrightarrow 0.$$

This is just the inverse image of the Tate central extension of $\mathcal{M}at$. □

Below we denote the Lie-* algebra \mathcal{L}^b by $\mathcal{L}_{\text{Tate}}$ and call it *the Tate central extension* of \mathcal{L} .

Remark: Note that for the complete curve X the short exact sequence of the D_X -modules that defines the extension $\mathcal{L}_{\text{Tate}}$ does not split, even if we forget about the Lie-* algebra structures.

3.4. **Local analog of the Tate extension.** Fix a point $x \in X$. Let $\hat{\mathcal{O}}_x$ (resp. $\hat{\mathcal{K}}_x$) be the completion of the local ring of the point x (resp. of the local field of the point x).

Consider the topological Lie algebra $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L})$ with the Lie subalgebra $DR^0(\text{Spec}(\hat{\mathcal{O}}_x), \mathcal{L})$.

3.4.1. **Lemma:**

- (i) $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L})$ is a *Tate Lie algebra*.
- (ii) The subalgebra $DR^0(\text{Spec}(\hat{\mathcal{O}}_x), \mathcal{L})$ is a *c-lattice* in $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L})$. □

In particular we have a one dimensional Lie algebra central extension

$$0 \longrightarrow \mathbb{C} \longrightarrow DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L}_{\text{Tate}}) \longrightarrow DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L}) \longrightarrow 0.$$

3.4.2. **Proposition:** The central extension $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L}_{\text{Tate}})$ coincides with the extension of the Tate Lie algebra $DR^0(\text{Spec}(\hat{\mathcal{K}}_x), \mathcal{L})$ with the help of the *critical cocycle* (see e.g. [Ar2], 4.3.3). \square

4. DG LIE ALGEBROIDS IN THE CATEGORY OF D_X -MODULES.

Recall that $\text{mod-}D_X$ is a symmetric tensor category. Below we use this structure to mimic the ordinary definition of a Lie algebroid in the category of vector spaces.

4.1. **Definition:** Let \mathcal{R} be a $\overset{!}{\otimes}$ -commutative algebra in $\text{mod-}D_X$, and let \mathcal{A} be a \mathcal{R} -module (so we have a D-module map $\mathcal{R} \overset{!}{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$ providing the structure) carrying a Lie algebra structure in $\text{mod-}D_X$ (i.e. a Lie bracket map $[\cdot, \cdot] : \Lambda^2(\mathcal{A}) \longrightarrow \mathcal{A}$ satisfying the Jacobi identity is given). Moreover suppose that \mathcal{A} acts on \mathcal{R} by derivations (i.e. we have a D-module map $\mathcal{A} \overset{!}{\otimes} \mathcal{R} \longrightarrow \mathcal{R}$ such that $[a, rb] = a(r)b + r[a, b]$ is satisfied for any sections $a, b \in \Gamma(X, \mathcal{A})$ and $r \in \Gamma(X, \mathcal{R})$).

We call the above data *the Lie algebroid* in the category $\text{mod-}D_X$ over the $\overset{!}{\otimes}$ -commutative algebra \mathcal{R} .

From now on we assume that all the appearing \mathcal{R} -algebroids are locally free as \mathcal{R} -modules.

4.1.1. By definition a *right module* (resp. a *left module*) over a Lie algebroid $(\mathcal{A}, \mathcal{R})$ on X is a sheaf of \mathcal{R} -modules \mathcal{M} with the Lie action of \mathcal{A} satisfying the constraint $(rm) \cdot a = r(m \cdot a) - (a(r))m$ (resp. $a \cdot (rm) = r(a \cdot m) + (a(r)) \cdot m$) for any sections a of \mathcal{A} , r of \mathcal{R} and m of \mathcal{M} .

4.1.2. Recall that the *universal enveloping algebra* for a \mathcal{R} -Lie algebroid \mathcal{L} in $\text{mod-}D_X$ is defined in the same way it is done for Lie algebroids over vector spaces: we take the free algebra in the category $\text{mod-}D_X$ generated by \mathcal{L} and take its quotient by the obvious ideal of relations including the one expressing the action of \mathcal{L} on \mathcal{R} by derivations. We denote the obtained associative algebra in $\text{mod-}D_X$ by $\mathcal{U}_{\mathcal{R}}(\mathcal{L})$.

4.2. **Homological Chevalley complex for a right module over a Lie algebroid in $\text{mod-}D_X$.** For a *right* \mathcal{A} -module \mathcal{M} consider the graded D_X -module on $C^\bullet(\mathcal{A}, \mathcal{M})$ as follows:

$$C^\bullet(\mathcal{A}, \mathcal{M}) = \bigoplus_k C^k(\mathcal{A}, \mathcal{M}), \quad C^k(\mathcal{A}, \mathcal{M}) = \mathcal{M} \otimes_{\mathcal{R}} (\Lambda_{\mathcal{R}}^{-k}(\mathcal{A})).$$

We endow the graded vector space with the differential as follows: For sections a_1, \dots, a_p of \mathcal{A} and m of \mathcal{M} we put

$$\begin{aligned} d(m \otimes a_1 \wedge \dots \wedge a_p) &= \sum_i (-1)^i m \cdot a_i \otimes a_1 \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge a_p \\ &+ \sum_{i < j} (-1)^{i+j} m \otimes [a_i, a_j] \wedge a_1 \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge \hat{a}_j \wedge \dots \wedge a_p. \end{aligned}$$

Lemma:

- (i) The differential in the complex is well defined.
- (ii) The differential satisfies $d^2 = 0$.

Proof. (i) Let us perform a calculation showing that the differential $d : C^{-k} \rightarrow C^{-k+1}$ is well defined for $k = 2$, the general case is quite similar. We have

$$\begin{aligned}
d(m \otimes r a_1 \wedge a_2) &= m \cdot (r a_1) \otimes a_2 - m \cdot a_2 \otimes r a_1 + m \otimes [a_2, r a_1] \\
&= m \otimes (a_2(r)) a_1 + m \otimes r [a_2, a_1] - r(m \cdot a_2) \otimes a_1 + m \cdot (r a_1) \otimes a_2 \\
&= (a_2(r)) m \otimes a_1 + r m \otimes [a_2, a_1] - (r m) \cdot a_2 \otimes a_1 - a_2(r) m \otimes a_1 + (r m) \cdot a_1 \otimes a_2 \\
&= (r m) \cdot a_1 \otimes a_2 - (r m) \cdot a_2 \otimes a_1 + r m \otimes [a_2, a_1] = d(r m \otimes a_1 \wedge a_2).
\end{aligned}$$

The general case is quite similar.

(ii) This is the usual calculation in the Chevalley complex. \square

Remark: We have shown above that there exists a *homological* Chevalley complex for a *right* \mathcal{A} -module \mathcal{M} . A very similar calculation proves the existence of the *cohomological* Chevalley complex for a *left* \mathcal{A} -module \mathcal{N} , of the size $\underline{\text{Hom}}_{\mathcal{R}}(\Lambda_{\mathcal{R}}^{\bullet}(\mathcal{A}), \mathcal{N})$.

4.2.1. In fact both the homological and the cohomological versions of the Chevalley complex for a Lie algebroid appear naturally “in a coordinate-free way” as a result of the following construction.

Consider the tautological *left* \mathcal{A} -module $\underline{\mathcal{R}}$. We construct its standard projective resolution in the way it is usually done for modules over Lie algebras.

Namely consider the complex

$$\text{Stan}^{\bullet}(\mathcal{A}, \underline{\mathcal{R}}) := \mathcal{U}_{\mathcal{R}}(\mathcal{A}) \otimes_{\mathcal{R}} \Lambda_{\mathcal{R}}^{\bullet}(\mathcal{A}) \otimes_{\mathcal{R}} \underline{\mathcal{R}}$$

with the standard differential. Note that the differential uses the right CA -module structure on $\mathcal{U}_{\mathcal{R}}(\mathcal{A})$.

Lemma:

- (i) The complex of D_X -modules $\text{Stan}^{\bullet}(\mathcal{A}, \underline{\mathcal{R}})$ is well-defined.
- (ii) $\text{Stan}^{\bullet}(\mathcal{A}, \underline{\mathcal{R}})$ is a complex of \mathcal{A} -modules.
- (iii) The homological Chevalley complex $C^{\bullet}(\mathcal{A}, \mathcal{M})$ for a right \mathcal{A} -module \mathcal{M} is isomorphic to $\mathcal{M} \otimes_{\mathcal{A}} \text{Stan}^{\bullet}(\mathcal{A}, \underline{\mathcal{R}})$.
- (iv) The cohomological Chevalley complex $\underline{\text{Hom}}_{\mathcal{R}}(\Lambda_{\mathcal{R}}^{\bullet}(\mathcal{A}), \mathcal{N})$ for a left \mathcal{A} -module \mathcal{N} is isomorphic to $\underline{\text{Hom}}_{\mathcal{A}}(\text{Stan}^{\bullet}(\mathcal{A}, \underline{\mathcal{R}}), \mathcal{N})$. \square

4.3. **DG Lie algebroids in D_X -modules.** Now let $\mathcal{R}^{\bullet} = \oplus_k \mathcal{R}^k$ be a graded supercommutative \otimes -algebra on X and let $\mathcal{A}^{\bullet} = \oplus \mathcal{A}^k$ be a graded super Lie algebroid over \mathcal{R} . Suppose also there are an odd derivation $d_{\mathcal{R}}$ on \mathcal{R}^{\bullet} of degree 1 and an odd derivation $d_{\mathcal{A}}$ of the Lie superalgebra \mathcal{A}^{\bullet} also of degree 1 satisfying Leibnitz rule with respect one to another. Moreover both of them satisfy $d^2 = 0$.

4.3.1. **Definition:** The data $(\mathcal{A}^{\bullet}, \mathcal{R}^{\bullet}, d_{\mathcal{A}}, d_{\mathcal{R}})$ are called the differential graded Lie algebroid in the category $\text{mod-}D_X$ or, for short, a DG Lie algebroid on X .

The notion of a left (resp. right) DG-module over a DG algebroid in $\text{mod-}D_X$ is a natural combination of the previous definitions and we do not spell it out explicitly. The category of left (resp. right) DG-modules over a DG Lie algebroid $\mathcal{A} = \mathcal{A}_X$ is denoted by $DG\text{-}\mathcal{A}^{\bullet}\text{-mod}$ (resp. $DG\text{-mod-}\mathcal{A}^{\bullet}$).

4.4. Homological Chevalley complex for a DG Lie algebroid in $\text{mod-}D_X$.

Now we sort of add a second differential on the homological Chevalley complex given in 4.2. For $\mathcal{M}^\bullet \in DG\text{-mod-}\mathcal{A}^\bullet$ consider the bigraded vector space $C^{\bullet\bullet}(\mathcal{A}^\bullet, \mathcal{M}^\bullet)$ as follows: $C^{\bullet\bullet}(\mathcal{A}, \mathcal{M}^\bullet) = \mathcal{M}^\bullet \otimes_{\mathcal{R}^\bullet} (\Lambda_{\mathcal{R}^\bullet}^\bullet(\mathcal{A}^\bullet))$, here the first grading comes from the number of wedges in the exterior product and

$$C^{\bullet k}(\mathcal{A}^\bullet, \mathcal{M}^\bullet) = (\mathcal{M}^\bullet \otimes_{\mathcal{R}^\bullet} (\Lambda_{\mathcal{R}^\bullet}^\bullet(\mathcal{A}^\bullet)))^k$$

in the graded tensor product sense.

Consider the two differentials on the bigraded vector space. The first one of the grading $(1, 0)$ is the usual Chevalley differential like in 4.2. The second differential of the grading $(0, 1)$ is provided by the differentials on \mathcal{M}^\bullet and $\Lambda_{\mathcal{R}^\bullet}^\bullet(\mathcal{A}^\bullet)$.

Consider the total grading on the bigraded space and the total differential on it.

4.4.1. **Lemma:** The differential $d_1 + d_2$ is well defined and its square equals zero. \square

5. SEMIINFINITE COHOMOLOGY VIA DG LIE ALGEBROIDS IN $\text{mod-}D_X$.

In this section we show that the standard complex for the computation of semi-infinite cohomology of a chiral module over a Lie- * algebra coincides with the homological Chevalley complex of the form 4.4 for a certain DG Lie algebroid in the category $\text{mod-}D_X$ and a certain right module over it.

5.1. **Construction of the Lie algebroid in $\text{mod-}D_X$.** Fix a Lie- * algebra \mathcal{L} on X . As before, we suppose that it is locally free and finitely generated over D_X .

The construction will be local and we can assume that X is affine.

Consider the completion of the $D_{X \times X}$ -module $\Omega \boxtimes \mathcal{L}$ along the diagonal. We denote this D-module by $\mathcal{L}_{\hat{\Delta}}$. One can view the D-module $\mathcal{L}_{\hat{\Delta}}$ as the restriction of $\Omega \boxtimes \mathcal{L}$ to the “family of formal discs” parametrized by the diagonal.

Consider also the D-module $j_*(\mathcal{O}_U) \otimes_{\mathcal{O}_{X \times X}} \mathcal{L}_{\hat{\Delta}}$ denoted by $\mathcal{L}_{\tilde{\Delta}}$. One can view the D-module $\mathcal{L}_{\tilde{\Delta}}$ as the restriction of $\Omega \boxtimes \mathcal{L}$ to the “family of punctured formal discs” parametrized by the diagonal.

5.1.1. **Lemma:** We have the short exact sequence of the d-modules

$$0 \longrightarrow \mathcal{L}_{\hat{\Delta}} \longrightarrow \mathcal{L}_{\tilde{\Delta}} \longrightarrow \Delta_!(\mathcal{L}) \longrightarrow 0 \quad \square$$

Now we take the D-module direct images $\mathcal{B} := p_{1*}(\mathcal{L}_{\hat{\Delta}})$ and $\mathcal{G} := p_{1*}(\mathcal{L}_{\tilde{\Delta}})$ on X (here p_1 is the projection $(x, y) \in X \times X \longrightarrow x \in X$). There is an obvious map $\mathcal{B} \longrightarrow \mathcal{G}$ and from the fact that \mathcal{L} is a Lie- * algebra we infer that both \mathcal{B} and \mathcal{G} are Lie algebras in the category of right D-modules on X .

Note that the stalk of the D-module \mathcal{B} (resp. of \mathcal{G}) at a point $x \in X$ equals $DR^0(\hat{\mathcal{O}}_x, \mathcal{L})$ (resp. $DR^0(\hat{\mathcal{K}}_x, \mathcal{L})$), where $\hat{\mathcal{O}}_x$ (resp. $\hat{\mathcal{K}}_x$) denotes the spectrum of the completed local ring (resp. the completed local field) at x . We abuse some notation here.

Lemma: There exists a short exact sequence of D-modules on X as follows:

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{G} \longrightarrow \mathcal{L} \longrightarrow 0.$$

Proof. Consider the short exact sequence of D-modules from the previous Lemma. Now take the (D-module) direct image of the exact sequence under p_1 . It is left to the reader that the sequence obtained as a result coincides with the one we need. \square

We choose the basic $\overset{!}{\otimes}$ -supercommutative algebra for our algebroid to be $\mathcal{R}^\bullet(\mathcal{L}) = \Lambda^\bullet(\mathcal{L}^*)$.

5.1.2. Lemma:

- (i) The Lie algebra in D-modules \mathcal{B} acts on \mathcal{L} .
- (ii) The Lie algebra in D-modules \mathcal{B} acts on the $\overset{!}{\otimes}$ -supercommutative algebra $\mathcal{R}^\bullet(\mathcal{L})$ by derivations.

Proof. Note that the second assertion of the Lemma follows from the first one since we can extend the action from the generators of $\mathcal{R}^\bullet(\mathcal{L})$ to the whole algebra by the Leibnitz rule.

Now we construct the action map for (i) explicitly. We have the following sequence of morphisms of D-modules

$$\begin{aligned} p_{1*}(\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} \mathcal{L}^* &\longrightarrow p_{1*}p_1^* \left(p_{1*}(\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} \mathcal{L}^* \right) \longrightarrow p_{1*} \left((\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} p_1^*(\mathcal{L}^*) \right) \\ &\xrightarrow{\sim} p_{1*}((\mathcal{L}^* \boxtimes \mathcal{L})_{\hat{\Delta}}) \xrightarrow{\mu} p_{1*}(\Delta_!(\mathcal{L}^*)) \xrightarrow{\sim} \mathcal{L}^*. \end{aligned}$$

Here $(\mathcal{L}^* \boxtimes \mathcal{L})_{\hat{\Delta}}$ denotes the completion of the D-module $\mathcal{L}^* \boxtimes \mathcal{L}$ along the diagonal in $X \times X$.

Note that the D-module $\Delta_!(\mathcal{L}^*)$ is locally finite over the ideal of the diagonal. Thus the completion of the map $\mathcal{L}^* \boxtimes \mathcal{L} \longrightarrow \Delta_!(\mathcal{L}^*)$ is well defined.

It is left to the reader that the composition of the above maps provides the action of the Lie algebra \mathcal{B} on \mathcal{L}^* . \square

Corollary: The graded D-module $\mathcal{A}^\bullet(\mathcal{L}) := \mathcal{B} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L})$ carries a natural structure of a Lie algebroid over the $\overset{!}{\otimes}$ -supercommutative algebra $\mathcal{R}^\bullet(\mathcal{L})$. \square

5.2. Construction of the differential on $\mathcal{A}^\bullet(\mathcal{L})$. Note that the differential $d_{\mathcal{R}}$ on the $\overset{!}{\otimes}$ -supercommutative algebra $\mathcal{R}^\bullet(\mathcal{L})$ is already constructed. Moreover it remains to construct the component of the differential on $\mathcal{B} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) = \mathcal{B} \overset{!}{\otimes} \Lambda^\bullet(\mathcal{L}^*)$ as follows: $d_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B} \overset{!}{\otimes} \mathcal{L}^*$. After that the differential on the whole Lie algebroid is obtained from $d_{\mathcal{R}} + d_{\mathcal{B}}$ by the Leibnitz rule.

Now by Lemma 2.1.2(i) we rewrite the map in question as $\mathcal{B} \boxtimes \mathcal{L} \longrightarrow \Delta_!(\mathcal{B})$ or

$$(p_{1*}(\mathcal{L}_{\hat{\Delta}})) \boxtimes \mathcal{L} \longrightarrow \Delta_!(p_{1*}(\mathcal{L}_{\hat{\Delta}})).$$

Note that $DR^0(\mathcal{L})$ acts naturally on every stalk of $p_{1*}(\mathcal{L}_{\hat{\Delta}})$ at any point $x \in X$. Recall that the stalk equals $DR^0(\hat{\mathcal{O}}_x, \mathcal{L})$. Thus the sheaf of Lie algebras acts on it acts on $(p_{1*}(\mathcal{L}_{\hat{\Delta}}))$.

5.2.1. Proposition: The above action is given by differential operators, i.e. it lifts to the required morphism $\mathcal{B} \boxtimes \mathcal{L} \longrightarrow \Delta_!(\mathcal{B})$.

Proof. This is a local assertion. It is left to the reader to check it. \square

Corollary: The D-module $\mathcal{A}^\bullet(\mathcal{L}) := \mathcal{B} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L})$ carries a natural structure of a DG Lie algebroid over the $\overset{!}{\otimes}$ -supercommutative DG algebra $\mathcal{R}^\bullet(\mathcal{L})$.

5.2.2. **Remark:** In particular the D-module $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}) := p_{1*}((\mathcal{L}_{\text{Tate}})_{\hat{\Delta}}) \overset{!}{\otimes} \Lambda^\bullet(\mathcal{L}^*)$ is a DG Lie algebroid over $\mathcal{R}^\bullet(\mathcal{L})$. This follows from the obvious fact that $\underline{\text{Hom}}_{D_X}(\mathcal{L}, D_X \otimes \Omega_X) = \underline{\text{Hom}}_{D_X}(\mathcal{L}_{\text{Tate}}, D_X \otimes \Omega_X)$.

5.3. **Construction of the left $\mathcal{A}^\bullet(\mathcal{L})$ DG-module for a chiral \mathcal{L} -module.** Note that the Lie algebra in the category of D-modules $\mathcal{G} = p_{1*}(\mathcal{L}_{\hat{\Delta}})$ acts naturally on an arbitrary chiral \mathcal{L} -module \mathcal{M} as follows:

$$\begin{aligned} p_{1*}(\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} \mathcal{M} &\longrightarrow p_{1*}\left((\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} p_1^*(\mathcal{M})\right) \longrightarrow p_{1*}\left((\mathcal{L}_{\hat{\Delta}}) \overset{!}{\otimes} (\mathcal{M} \boxtimes \Omega)\right) \\ &\longrightarrow p_{1*}((\mathcal{M} \boxtimes \mathcal{L})_{\hat{\Delta}}) \xrightarrow{\mu} p_{1*}\Delta_!(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}. \end{aligned}$$

Here $(\mathcal{M} \boxtimes \mathcal{L})_{\hat{\Delta}}$ denotes the completion of the D-module $j_*j^*(\mathcal{M} \boxtimes \mathcal{L})$ along the diagonal in $X \times X$.

Note that the D-module $\Delta_!(\mathcal{M})$ is locally finite over the ideal of the diagonal. Thus the completion of the map $j_*j^*(\mathcal{M} \boxtimes \mathcal{L}) \longrightarrow \Delta_!(\mathcal{M})$ is well defined.

5.3.1. For a chiral \mathcal{L} -module \mathcal{M} consider it as a Lie- * module and recall its naive de Rham complex $C^\bullet(\mathcal{L}, \mathcal{M}) = \Lambda^\bullet(\mathcal{L}^*) \overset{!}{\otimes} \mathcal{M}$.

Lemma: The complex $C^\bullet(\mathcal{L}, \mathcal{M})$ has a natural structure of a *left* DG-module over the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L})$.

Proof. The statement of the Lemma follows from the existence of the \mathcal{G} -module structure on \mathcal{M} introduced in the beginning of the present subsection. \square

Here we come to the crucial point explaining the phenomenon of the Tate extension in the semiinfinite cohomology of Lie- * algebras. What we would like to do is to consider the homological Chevalley complex of the DG Lie algebroid $(\mathcal{A}^\bullet(\mathcal{L}), \mathcal{R}^\bullet(\mathcal{L}), \dots)$ with coefficients in $C^\bullet(\mathcal{L}, \mathcal{M})$. Yet there is no naive way to do it. Somehow we have to make $\mathcal{M} \otimes \Lambda^\bullet(\mathcal{L}^*)$ into a *right* DG-module over our DG Lie algebroid in the category of D_X -modules.

Remark: Mimicking the Tate Lie algebra case we could consider the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L}) \oplus \mathcal{R}^\bullet(\mathcal{L}) \otimes \mathbf{1}$, then construct its antipode α not commuting with the differential. However the point is that the obtained DG Lie algebroid in the category $\text{mod-}D_X$ *does not come from a central extension of our Lie- * algebra \mathcal{L}* . Thus there is no way to construct a left DG-module over $(\mathcal{A}^\bullet(\mathcal{L}) \oplus \mathcal{R}^\bullet(\mathcal{L}) \otimes \mathbf{1})^{\text{opp}}$ (with the differential twisted by the antipode) starting from a chiral module either over \mathcal{L} or over some its central extension. Instead one should act as follows.

5.4. **Antipode map for the Tate extension of $\mathcal{A}^\bullet(\mathcal{L})$.** Consider the (Tate) extension of Lie algebras in D-modules on X :

$$0 \longrightarrow \Omega_X \longrightarrow \mathcal{B}_{\text{Tate}} \longrightarrow \mathcal{B} \longrightarrow 0.$$

Here $\mathcal{B}_{\text{Tate}}$ denotes $\mathcal{B}(\mathcal{L}_{\text{Tate}})$. We have also the corresponding extension of DG Lie algebroids on X :

$$0 \longrightarrow \Omega_X \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) \longrightarrow \mathcal{B}_{\text{Tate}} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) \longrightarrow \mathcal{B} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) \longrightarrow 0.$$

Denote $\mathcal{B}_{\text{Tate}} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L})$ by $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})$.

We will need also the universal enveloping algebras of the DG-Lie $\mathcal{R}^\bullet(\mathcal{L})$ -algebroids $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})$ and $\mathcal{A}^\bullet(\mathcal{L})$. Keeping the notation from the previous section we denote these associative algebras in $\text{mod-}D_X$ by $\tilde{\mathcal{U}}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$ and $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}))$ respectively. Let $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$ be the quotient of $\tilde{\mathcal{U}}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$ by the ideal generated by the relation $\mathcal{R}^\bullet(\mathcal{L}) \overset{!}{\otimes} (\Omega_X \otimes \mathbf{1}) = \mathcal{R}^\bullet(\mathcal{L})$. Here the LHS of the equality is the kernel of the DG Lie algebroid extension map while the RHS is the unit $\mathcal{R}^\bullet(\mathcal{L}) \subset \tilde{\mathcal{U}}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$.

Note that with the differentials forgotten the algebras $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$ and $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}))$ are isomorphic.

5.4.1. We introduce an antipode map $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}) \xrightarrow{\sim} \mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})^{\text{opp}}$ as follows. Set $\alpha(b \otimes r) = -b \otimes r + \text{co-ad}_b(r) \otimes \mathbf{1}$. Here b is a section of $\mathcal{B}_{\text{Tate}}$, r is a section of $\mathcal{R}^\bullet(\mathcal{L})$ and $\mathbf{1}$ is the generating section of Ω_X .

Note that the antipode does not necessarily commute with the differential on $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})$.

Lemma:

- (i) α is well defined as a antipode of a Lie superalgebra in the category of D_X -modules $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})$ (with the differential forgotten).
- (ii) When restricted to any open affine subset $\overset{\circ}{X} \subset X$ the DG Lie superalgebra in the category of D_X -modules $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})^{\text{opp}}|_{\overset{\circ}{X}}$ with the differential $\alpha \circ d_{\mathcal{A}^\bullet} \circ \alpha^{-1}$ is isomorphic to the DG Lie superalgebra in the category of D_X -modules $\mathcal{A}^\bullet(\mathcal{L} \oplus \Omega_X)|_{\overset{\circ}{X}}$. Here $\mathcal{L} \oplus \Omega_X$ denotes the trivial central extension.

Proof. Both statements of the Lemma follow from the corresponding local statements presented in [Ar2], Proposition 4.3.4. \square

Here we come to another difference with the Tate Lie algebra semiinfinite cohomology case. While previous Lemma states that when restricted to an open affine subset the complex of D-modules $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})^{\text{opp}}$ is isomorphic to $\mathcal{A}^\bullet(\mathcal{L}) \oplus \mathcal{R}^\bullet(\mathcal{L}) \overset{!}{\otimes} (\Omega_X \otimes \mathbf{1})$, still possibly the short exact sequence of D-modules on the complete curve

$$0 \longrightarrow \Omega_X \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) \longrightarrow (\mathcal{B}_{\text{Tate}} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}))^{\text{opp}} \longrightarrow \mathcal{B} \overset{!}{\otimes} \mathcal{R}^\bullet(\mathcal{L}) \longrightarrow 0$$

does not split.

That is where we use the universal enveloping algebras of our DG Lie algebroids. Extend the antipode α to $\tilde{\mathcal{U}}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$.

5.4.2. **Proposition:**

- (i) α descends to the antipode of $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))$.
- (ii) The DG-algebra $\mathcal{U}_{\mathcal{R}}(\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}}))^{\text{opp}}$ with the differential twisted by the antipode α is isomorphic to the DG-algebra $\mathcal{U}_{\mathcal{R}}(\mathcal{A}(\mathcal{L}))$.

Proof. Follows from local calculations in the Tate Lie algebra semiinfinite cohomology case (see [Ar2]).

Corollary: Any *left* DG-module over the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L}_{\text{Tate}})$ on which the center $\Omega_X \otimes \mathbf{1}$ acts by unity becomes a *right* DG-module over the DG Lie algebroid $\mathcal{A}^\bullet(\mathcal{L})$. \square

5.5. Standard semiinfinite complex for a chiral module over Tate extension of the Lie- \ast algebra \mathcal{L} . For a chiral $\mathcal{L}_{\text{Tate}}$ -module \mathcal{M} such that the center $\Omega_X \otimes \mathbf{1}$ acts on it by unity consider the naive de Rham complex of the module $C^\bullet(\mathcal{L}_{\text{Tate}}, \mathcal{M}) = \mathcal{M} \otimes \Lambda(\mathcal{L}^*)$ as a *right* DG-module over the DG-Lie algebroid in the category of D_X -modules $(\mathcal{B} \otimes \mathcal{R}^\bullet(\mathcal{L}))$. The right DG-module structure is obtained using the antipode construction from the previous subsection.

5.5.1. Definition: We call the homological Chevalley complex of the DG Lie algebroid $\left((\mathcal{B} \otimes \mathcal{R}^\bullet(\mathcal{L})), \mathcal{R}^\bullet(\mathcal{L}), \dots \right)$ with coefficients in the right DG-module $C^\bullet(\mathcal{L}_{\text{Tate}}, \mathcal{M})$ the *standard semiinfinite complex* for the chiral module \mathcal{M} over the Lie- \ast algebra $\mathcal{L}_{\text{Tate}}$ and denote it by $C^{\frac{\infty}{2}}(\mathcal{L}_{\text{Tate}}, \mathcal{M})$.

Remark: Note that as the D-module on X the constructed complex looks as follows:

$$\begin{aligned} C^{\frac{\infty}{2}}(\mathcal{L}_{\text{Tate}}, \mathcal{M}) &= \Lambda_{\mathcal{R}^\bullet}^\bullet \left(p_{1*}((\mathcal{L})_{\hat{\Delta}} \otimes \mathcal{R}^\bullet(\mathcal{L})) \right) \otimes_{\mathcal{R}^\bullet} \left(\mathcal{M} \otimes \Lambda^\bullet(\mathcal{L}^*) \right) \\ &= \Lambda^\bullet(p_{1*}((\mathcal{L})_{\hat{\Delta}}) \otimes \Lambda^\bullet(\mathcal{L}^*) \otimes \mathcal{M}. \end{aligned}$$

Here as before p_1 denotes the projection $X \times X \longrightarrow X$, $(x, y) \mapsto x$.

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